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THE FIELD OF THE POINT SOURCE OF INTERNAL WAVES IN A HALF-SPACE WITH A VARIABLE BRUNT-VAISALA FREQUENCY*

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Green's function is constructed for the equation of the internal waves in the half-space $z>0$ with a square of the Brunt-Vaisala frequency which is linear with respect to z.

1. Formulation of the problem. The generalized solution $\Gamma(t, \sqrt{x^2+y^2}, z, z_0)$ of the equation

$$L\Gamma = \left(\frac{\partial^2}{\partial t^2}\Delta + B^2 z\Delta_h\right)\Gamma = 0 \tag{1.1}$$
$$\Delta_h = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \Delta = \Delta_h + \frac{\partial^2}{\partial z^2}$$

with the initial and boundary conditions

$$\Gamma = 0, \quad \partial\Delta\Gamma/\partial t = \delta(x)\,\delta(y)\,\delta(z-z_0) \quad (t=0) \tag{1.2}$$
$$\Gamma = 0 \quad (z=0) \tag{1.3}$$

is considered in the half-space $z>0$.

It is obvious that, when this function is extended to zero at $t<0$, it satisfies the equation

$$L\Gamma = \delta(t)\,\delta(x)\,\delta(y)\,\delta(z-z_0) \tag{1.4}$$

that is, it is Green's function for the internal wave equation when the square of the buoyancy frequency $N^2(z) = B^2 z$; $B = \text{const}$.

The approximate expression for Γ has the form /1/ (J_0 is a Bessel function and v is an Airy function)

$$G(t, r, z, z_0) = -\frac{1}{\pi^2 B}\int_0^\infty \sigma^{1/3} d\sigma \int_0^\infty U\, d\omega \tag{1.5}$$

$$U = v(\sigma^{2/3}(\omega^2 - z))\, v(\sigma^{2/3}(\omega^2 - z_0)) \sin B\omega t J_0(\sigma\omega r) \tag{1.6}$$

Since $v(\xi)$ satisfies Airy's equation, it can be shown that the function U and, together with it, also G is an exact solution of Eq.(1.1). The approximate nature of the function G manifests itself in the fact that it does not satisfy the boundary condition (1.3) while the second condition in (1.2) is satisfied with an accuracy up to a smooth term Ψ:

$$\frac{\partial}{\partial t}\Delta G\,|_{t=0} = \delta(x)\,\delta(y)\,\delta(z-z_0) + \Psi(r, z, z_0)$$

It is natural to assume that the exact Green's function also has the form of (1.5), (1.6) where, however, the product of the Airy functions should be replaced by any other combination of solutions of Airy's equation for the same arguments. The condition regarding the symmetry of Γ with respect to z, z_0, the boundary conditions (1.3) and, finally, the requirement that

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the integrand in (1.5) should have the same asymptotic form when $\sigma\to\infty$ and $\omega<z, z_0$ as in (1.6) (this condition is found to be essential if the initial conditions are to be satisfied) enabled us to guess the form of the function U and to write out the following expression:

$$\Gamma=\frac{1}{4\pi^2 B}\int_0^\infty \sigma^{1/3}\,d\sigma\int_0^\infty \sin B\omega t J_0(\sigma\omega r)\,H(\sigma^{2/3}\omega^2)\,\Phi(z,\sigma,\omega)\,\Phi(z_0,\sigma,\omega)\,d\omega \tag{1.7}$$

$$\Phi(z,\sigma,\omega)=w_1(\sigma^{2/3}(\omega^2-z))\,w_2(\sigma^{2/3}\omega^2)-w_1(\sigma^{2/3}\omega^2)\,w_2(\sigma^{2/3}(\omega^2-z))$$

$$H(\xi)=[w_1(\xi)\,w_2(\xi)]^{-1},\quad w_{1,2}=u(\xi)\pm iv(\xi)$$

where $w_{1,2}$ are the first and second Airy functions. Integral representations and the principal properties of the functions w_1, w_2, u and v are given, for example, in /2, 3/.

The aim of the present paper was to prove formula (1.7) for Green's function and to determine its asymptotic behaviour as $t\to\infty$ as well as when t is fixed and $z\to z_0, r\to 0$. The integral representations of the products of Airy functions used in this paper are derived in paragraph 5.

2. Verification of the initial condition. The integrand in (1.2) is the solution of Eq.(1.1) and, hence, Γ, when $t>0$, is the generalized solution of this equation. It is obvious that Γ satisfies condition (1.3) and the first equation in (1.2) and it therefore suffices to verify that the second condition in (1.2) is satisfied, that is, that

$$\frac{\partial}{\partial t}\Gamma\bigg|_{t=0}=\frac{1}{4\pi}\left[(r^2+(z+z_0)^2)^{-1/2}-(r^2+(z-z_0))^{-1/2}\right] \tag{2.1}$$

If the variables $p=\sigma^{1/3}\omega$ and $q=\sigma\omega$ are substituted into the integral which expresses $\partial\Gamma/\partial t$ when $t=0$, formula (2.1) reduces to the following relationships:

$$I_1=\int_0^\infty H(p^2)\,F(p,qz)\,F(p,qz_0)\,dp=\frac{2}{3}\pi\exp[-q\,|\,z+z_0\,|)]- \tag{2.2}$$

$$4\int_0^\infty v\left(p^2-\frac{qz}{p}\right)v\left(p^2-\frac{qz_0}{p}\right)dp$$

$$F(p,qz)=w_1(p^2-qz/p)\,w_2(p^2)-w_1(p^2)\,w_2(p^2-qz/p)$$

$$I_2=\int_0^\infty J_0(qr)\,dq\int_{-\infty}^\infty v\left(p^2-\frac{qz}{p}\right)v\left(p^2-\frac{qz_0}{p}\right)dp=\frac{\pi}{6}\left[r^2+(z-z_0)^2\right]^{-1/2} \tag{2.3}$$

Let us outline the proof of these formulae. Taking account of the fact that $v(\xi)=(w_1(\xi)-w_2(\xi))/(2i)$, we write I_1 in the form $I_{11}+I_{12}$ where

$$I_{11}=-4\int_0^\infty v\left(p^2-\frac{qz}{p}\right)v\left(p^2-\frac{qz_0}{p}\right)dp \tag{2.4}$$

$$I_{12}=-4\lim_{M\to\infty}\operatorname{Im}\int_0^M w_2\left(p^2-\frac{qz}{p}\right)w_2\left(p^2-\frac{qz_0}{p}\right)\frac{v(p^2)}{w_2(p^2)}\,dp= \tag{2.5}$$

$$-4\lim_{M\to\infty}\operatorname{Im}\left[\int_0^{M\exp(\pi i/3)}+\int_{M\exp(\pi i/3)}^{M}\right]=-4\lim_{M\to\infty}[I_{13}+I_{14}]$$

We put $p=s\exp(\pi i/3)$ in the integral I_{13} and pass to integration with respect to s. Taking account of $(D_1 13)$ from /3/, we get

$$I_{13}=\int_0^M v\left(s^2+\frac{qz}{s}\right)v\left(s^2+\frac{qz_0}{s}\right)ds \tag{2.6}$$

In calculating I_{14}, we make use of the asymptotic behaviour of the integrand in (2.5) when $|\,p\,|\to\infty, 0\leqslant\arg p\leqslant\pi/3$ which follows from /3/ (pp.415-417). As a result, we get

$$I_{14}=-{}^1/_6\pi\exp[-q\,|\,z+z_0\,|]+O(M^{-1}) \tag{2.7}$$

By using (2.6) and (2.7) to calculate I_{12} and taking account of (2.4), we get (2.2).

In order to prove inequality (2.3), it is first necessary to integrate over the domain $-\infty<q<\infty, p>0$ (which does not change the value of the integral I_2), secondly, to make the substitution $q=p\xi$ and, thirdly, to make use of the integral representation (5.4) (paragraph 5) for the product of the Airy functions. By integrating the resulting integral with respect to p (taking account of the fact that $\operatorname{Im}\alpha>0$ and using formula 6.728 from /4/) and, then, with respect to ξ and α, we get (2.3).

Hence, formula (1.7) for Green's function is proved.

3. The asymptotic behaviour of Γ as $t \to \infty$. Let us write Γ in the form

$$\Gamma(t, r, z, z_0) = \frac{1}{4\pi^2 B} \int_0^\infty \sin B\omega t \Psi(\omega) \, d\omega \tag{3.1}$$

$$\Psi(\omega) = \int_0^\infty \sigma^{1/3} J_0(\sigma\omega r) H(\sigma^{2/3}\omega^2) \Phi(z, \sigma, \omega) \Phi(z_0, \sigma, \omega) \, d\omega \tag{3.2}$$

It can be seen from formula (3.1) that the asymptotic behaviour of Γ as $t \to \infty$ is determined, firstly, by the contribution of $f_0(t)$ to the integral (3.1) from the boundary of the interval of integration, and, secondly, by the contribution of $f_1(t)$ to (3.1) from the singular points of the function $\Psi(\omega)$:

$$\Gamma(t, r, z, z_0) \approx f_0(t) + f_1(t) \tag{3.3}$$

Let us show that, as $t \to \infty$, the function $f_0(t)$ tends to zero more rapidly than t^{-1}. In order to do this, it is sufficient to verify that

$$\lim_{\omega \to 0} \Psi(\omega) = 0 \tag{3.4}$$

In the calculation of this limit, one may put $\sigma^{2/3}\omega^2 = 0$, that is, it is possible to consider the integral

$$\Psi_1(\omega) = \int_0^\infty \sigma^{1/3} J_0(\sigma\omega r) \Phi(z, \sigma, 0) \Phi(z_0, \sigma, 0) \, d\sigma \tag{3.5}$$

We shall omit a rigorous proof of this assertion because of its length and we shall merely note here that, as $\omega \to 0$, $0 < \sigma < \omega^{-4/3}$ makes the main contribution to the integral (3.2) and within the limits of this interval, $\sigma^{2/3}\omega^2 \to 0$. Let us carry out the following transformations on the integral which has been written out. Firstly, we introduce a factor $\exp(-q\sigma)$ under the integral sign and we shall consider the limit of the resulting expression as $q \to 0$. Secondly, we shall make use of the integral representation /4/ (formula 8.44 (1)) for $J_0(\sigma\omega r)$. Thirdly and finally, use is made of the integral representations (5.1)-(5.3), in which we put $\alpha = \sigma^{1/3}\beta$ for the pairwise products of functions occurring in (3.5). By integrating with respect to σ and subsequently passing to the limit as $q \to 0$ and as $\omega \to 0$, we get the vanishing sum of the three multiple contour integrals, and (3.4) follows from this.

Let us now find the function $f_1(t)$ that is, the contribution to the asymptotic form of the integral (3.1) from the singular points of the function $\Psi(\omega)$. An analogous problem has been considered in /1/ for an approximate Green's function $G(t, r, z, z_0)$ which can be written in a form similar to (3.1):

$$G = -\int_0^\infty \sin B\omega t F(\omega) \, d\omega \tag{3.6}$$

$$F(\omega) = \frac{1}{\pi^2 B} \int_0^\infty \sigma^{1/3} J_0(\sigma\omega r) v(\sigma^{2/3}(\omega^2 - z)) v(\sigma^{2/3}(\omega^2 - z_0)) \, d\sigma \tag{3.7}$$

The function $\Psi(\omega)$ has the same singular points as $F(\omega)$. Actually, by putting $w_{1,2}(\xi) = u(\xi) \pm iv(\xi)$ and using the well-known asymptotic forms of the functions $u(\xi)$ and $v(\xi)$ as $\xi \to \infty$ it is possible to write the difference $F(\omega) - \Psi(\omega)$ (where $\Psi(\omega)$ has the form of (3.2)) in the form of an integral with respect to σ, the integrand in which exponentially decays as $\sigma \to \infty$ uniformly with respect to ω (when ω is bounded from below). Hence, $F(\omega) - \Psi(\omega)$ is an analytical function of ω when $\omega > 0$ and it follows from this that the contribution from the singular points of $\Psi(\omega)$ to the asymptotic form of Γ is identical with the contribution from the singular points of $F(\omega)$ to the asymptotic form of G. By using the refining the results in paragraph 4 from /1/, we obtain that, as $t \to \infty$, the function Γ has the following asymptotic form:

when $r\sqrt{z_-} < {}^2/_3(z_+ - z_-)^{3/2}$

$$\Gamma(t, r, z, z_0) = o(t^{-1}) \tag{3.8}$$

when $r\sqrt{z_-} > {}^2/_3(z_+ - z_-)^{3/2}$

$$\Gamma(t, r, z, z_0) \approx (\omega_1 t)^{-1/2} A(r, z, z_0, \omega_1) \sin\left(\omega_1 t + \frac{\pi}{4}\right) + (\omega_2 t)^{-1/2} A(r, z, z_0, \omega_2) \sin\left(\omega_2 t - \frac{\pi}{4}\right) + O(t^{-3/2}) \tag{3.9}$$

Here $\omega_{1,2} = \omega_{1,2}(r, z, z_0)$ are the roots of the equation

$$r\omega_{1,2} = \frac{2}{3B^2}\left[(B^2z_+ - \omega_{1,2}^2)^{3/2} \mp (B^2z_- - \omega_{1,2}^2)^{3/2}\right]$$

$$z_- = \min(z, z_0), \quad z_+ = \max(z, z_0)$$

$$A(r, z, z_0, \omega) = (2\pi)^{-1/2}\left[(B^2z - \omega^2)(B^2z_0 - \omega^2)\right]^{-1/4}\left[\frac{1}{r\omega}\frac{\partial\omega}{\partial r}\right]^{1/2}$$

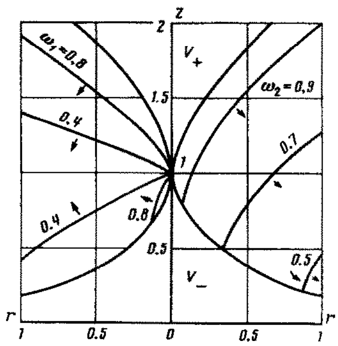

Fig.1

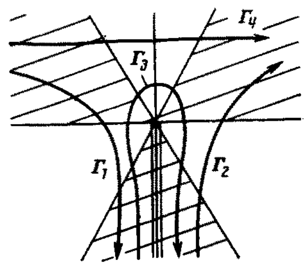

Fig.2

We will now comment on the results which have been obtained. The domain $r\sqrt{z_-} < {}^2/_3(z_+ - z_-)^{3/2}$ forms two funnels V_+ and V_- which emerge from the point $r = 0$; $z = z_0$ (see Fig.1, where $B = 1$, $z_0 = 1$). Within these funnels, Green's function does not oscillate and decays more rapidly than t^{-1}. Outside of the funnels, it decays as $t^{-1/2}$ and consists of two oscillating terms, the characteristic period of the oscillations of which tends to zero as t^{-1}. The crests of the waves of these terms are specified by the formulae

$$\omega_{1,2}(r, z, z_0) = (n \pm {}^1/_4)\,\pi/(Bt) \quad (n = 1, 2, \ldots) \tag{3.10}$$

that is, they coincide with the lines of the level of the functions ω_1 and ω_2. These level lines are shown in Fig.1: to the right of the axis $r = 0$ in the case of the function ω_2 and to the left of this axis in the case of ω_1. As t increases, the crests of each of the terms in (3.9) move towards decreasing $\omega_{1,2}$, that is, in the direction shown by the arrows in Fig.1.

Let us put $z_0 = N^2/B^2$ and make B tend to zero at fixed values of t, r, and $z - z_0$. Then, $\Gamma(t, r, z, z_0)$ will tend to Green's function $\Gamma_0(t, r, z - z_0)$ of the internal wave equation for a constant buoyancy frequency N. On passing to the limit in this manner, the funnels V_+ and V_- contract towards the $r = 0$ axis, the function $\omega_2(r, z, z_0)$ tends to N and ω_1 tends to $N|\cos\theta|$ (where $\operatorname{ctg}\theta = |z - z_0|/r$), while (3.9) passes into the well-known /5/ asymptotic form of the function Γ_0 as $t \to \infty$.

4. The asymptotic form of Γ when $r > 0, z \to z_0$.

When $N = \text{const}$, Green's function $\Gamma_0(t, r, z - z_0)$ has the form /5/

$$\Gamma_0 = \frac{-1}{2\pi^2\rho}\int\limits_{N\cos\theta}^{N}\frac{\sin\omega t\,d\omega}{\sqrt{N^2 - \omega^2}\sqrt{\omega^2 - N^2\cos^2\theta}}$$

$$\rho = (r^2 + (z - z_0)^2)^{1/2}, \quad \cos\theta = |z - z_0|/\rho$$

that is, it has a singularity as $\rho \to 0$. Let us show that Green's function which has been constructed above, Γ, has the same singularity as Γ_0 when $r \to 0$, $z \to z_0$ if the value of the Brunt-Vaisala frequency N on the horizon of the source $z = z_0$, N, is taken as $B\sqrt{z_0}$:

$$\lim_{\rho\to 0}\rho\Gamma = -\frac{1}{2\pi^2 B}\int\limits_{\sqrt{z_0}\cos\theta}^{\sqrt{z_0}}\frac{\sin B\omega t\,d\omega}{\sqrt{z_0 - \omega^2}\sqrt{\omega^2 - z_0\cos^2\theta}} \tag{4.1}$$

In order to prove (4.1), we note that the approximate Green's function G expressed by formulae (3.6) and (3.7) has the same singularity as Γ when $\rho \to 0$. Actually, this singularity is due to the poor convergence of the integrals expressing Γ and G when $\sigma, \omega \to \infty$ while their difference is described by an integral which converges absolutely and uniformly with respect to r, z, and z_0 when $\sigma, \omega \to \infty$. It is therefore sufficient to find the limit of ρG.

Let us put $r = \rho\sin\theta$; $z - z_0 = \rho\cos\theta$ and seek the limit of ρG as $\rho \to 0$ and θ is fixed. We use formulae (3.6) and (3.7) for G and formula (5.4) for the product of the Airy functions after which we carry out a change of variables $p = \sigma r$; $q = \alpha\sigma^{-1/3} r^{-1}$ and pass to the limit as $r \to 0$; $z \to z_0$. By integrating the resulting expressions with respect to q and then with respect to p, we get (4.1).

5. Integral representations for products of Airy functions.

Let us put

$$I_k = -\pi^{-1/2} e^{\pi i/4} \int_{\Gamma_k} \exp iF(\alpha) \frac{d\alpha}{\sqrt{\alpha}}$$

$$F(\alpha) = \alpha^3/12 + \alpha(p+q)/2 - (p-q)^2/(4\alpha)$$

where the contours Γ_k $(k = 1, 2, 3, 4)$ are shown in Fig.2. The following relations hold:

$$w_1(p)\, w_1(q) = I_1 \tag{5.1}$$

$$w_2(p)\, w_2(q) = I_2 \tag{5.2}$$

$$w_1(p)\, w_2(q) + w_2(p)\, w_1(q) = -I_3 \tag{5.3}$$

$$4v(p)\, v(q) = -I_4 \tag{5.4}$$

In order to prove (5.1)-(5.4), we note that each of the integrals I_k satisfies Airy's equation both with respect to the variable p as well as with respect to the variable q. In fact, by applying the operators $d^2/dp^2 - p$ and $d^2/dq^2 - q$ to I_k, we obtain the complete differential of the function $i\alpha^{-1/2} \exp iF(\alpha)$ under the integral sign. Allowing for the fact that the I_k are symmetrical with respect to q, we obtain

$$I_k = \gamma_k w_1(p)\, w_1(q) + \omega_k [w_1(p)\, w_2(q) + w_2(p)\, w_1(q)] + \delta_k w_2(p)\, w_2(q) \tag{5.5}$$

In order to find the constants γ_k, ω_k, δ_k, we put $p = -m^2$, $q = -n^2$, $m > n \gg 1$ and equate the asymptotic form of the right-hand side in (5.5) to the asymptotic form of the integrals I_k found by the method of steepest descent. As a result, we get formulae (5.1)-(5.4).

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